Thermal effects on chaotic directed transport

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We study a chaotic ratchet system under the influence of a thermal environment. By direct integration of the Lindblad equation we are able to analyze its behavior for a wide range of couplings with the environment, and for different finite temperatures. We observe that the enhancement of the classical and quantum currents due to temperature depend strongly on the specific properties of the system. This makes it difficult to extract universal behaviors. We have also found that there is an analogy between the effects of the classical thermal noise and those of the finite $\hbar$ size. These results open many possibilities for their testing and implementation in kicked Bose-Einstein condensates and cold atoms experiments.

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I. INTRODUCTION

The first theoretical constructions related with directed transport (transport phenomena in spatially periodic systems that are out of thermal equilibrium) were formulated in an early work by Feynman [1]. This opened a field whose relevance and activity has been increasing since then. The motivation is twofold, in the first place several fundamental questions have originated from these ideas and have been only partially answered [2]. In our opinion, some of the most important ones are related to the possibility of having a net transport that is compatible with the second law of thermodynamics or the Liouville theorem for Hamiltonian cases. The answers were formulated in the shape of necessary conditions on symmetries and sum rules [3,4]. Nowadays, the question of how quantum mechanical effects translate into these or other conditions seems also very interesting [5]. As a consequence of this (and several other investigations), the subject grew into a major new field of statistical physics. On the other hand, the wealth of possible applications has determined the emergence of a very broad area of research. Ratchets, generically defined as periodic systems having a dissipative dynamics associated with thermal noise and unbiased perturbations (driving them out of equilibrium), can be used to model a wide range of different phenomena. In order to give just a few examples we can mention molecular motors in biology [6], nanodevices (rectifiers, pumps, particle separators, molecular switches, and transistors) [7], and coupled Josephson junctions [8]. On the other hand, there is a great interest in the theoretical description and experimental implementations of cold atoms subjected to time-dependent standing waves of light. They play a central role in many studies of the quantum dynamics of nonlinear systems (dynamical localization, decoherence, quantum resonance, etc.) [9]. In particular, the so-called optical ratchets, i.e., directed transport of laser cooled atoms, have been successfully implemented in this sort of experiment [10].

The appearance of a net current can be classically explained by the necessary condition of breaking all spatiotemporal symmetries leading to momentum inversion [3]. This, along with Curie's principle (which essentially says that if symmetries do not forbid a given phenomenon, then it will manifest itself) indicates that transport should be present. As an example of this situation we can mention Hamiltonian systems (with necessarily mixed phase spaces) where a chaotic layer should be asymmetric [4]. In the more general dissipative case, chaotic attractors having this property are necessary [11]. It is usual that the same principle translates almost directly into the quantum domain [12], and very similar behaviors arise. But sometimes quantum mechanics introduces new effects [13] and the results depart from the classical ones. For example, the efficiency of a forced thermal quantum ratchet has been calculated in [14]. In that work the authors find that the quantum current is higher in comparison with the classical one at the lowest values of the temperature. As this parameter increases the discrepancies diminish and finally they become negligibly small.

Recently, a quantum chaotic dissipative ratchet has been introduced in [15]. In this example directed transport appears for particles under the influence of a pulsed asymmetric potential in the presence of a dissipative environment at zero temperature. The asymmetry of the quantum strange attractor is at the origin of the quantum current, in close analogy with what happens at the classical level. Indeed, this work provides with the parameters needed for a possible implementation using cold atoms in an optical lattice. For a somewhat similar dynamics, the case of weak coupling and low temperature has been studied in [16]. In the present work we extend the study of [15] for a wide range of couplings with the environment and different temperature values. We have verified that there is a strong dependence of the current behavior on the coupling strength. If we compare this with the results found in [14], for instance, we could not find a generic enhancement of the quantum current for finite temperatures. Instead of that, we could identify a close quantum-to-classical correspondence when considering thermal effects only at the classical level. In fact, we have found that the finite $\hbar$ effects on the quantum current are analogous to the influence of the classical thermal fluctuations on the classical transport.

In the following we describe the organization of this paper. In Sec. II we present our model for the system and for the environment, explaining the methods we have used to investigate the current behavior. In Sec. III we show the results where the roles of $\hbar$, the coupling strength and the temperature are analyzed in detail. Finally, in Sec. IV we summarize and point out our conclusions.

II. SYSTEM AND ENVIRONMENT

In this section we describe the approach used to model the system and the environment. We study the motion of a particle in a periodic kicked asymmetric potential given by

$$V(x,t) = k[\cos x + a/2 \cos(2x+\Phi)] \sum_{m=-\infty}^{+\infty} \delta(t-m\tau), \quad (1)$$

where τ is the kicking period, k is the strength of the kick, and a and Φ are parameters that allow us to introduce a spatial asymmetry [15]. The effects of the environment are taken into account by means of a velocity-dependent damping and thermal fluctuations. At the classical level, these ingredients are incorporated in the following map:

$$\bar{n} = \Gamma n + k[\sin x + a \sin(2x+\Phi)] + \xi,$$

$$\bar{x} = x + \tau\bar{n}. \quad (2)$$

In these expressions, n is the momentum variable conjugated to x and Γ is the dissipation parameter, with $0 \leqslant \Gamma \leqslant 1$. The thermal noise ξ is related to Γ, according to $\langle \xi^2 \rangle = 2(1-\Gamma)k_B T$, where k_B is the Boltzmann constant and T is the temperature, making the formulation consistent with the fluctuation-dissipation relationship. By performing the change of variables $\tau n \to p$, $\tau k \to K$, and $\tau\xi \to \tilde{\xi}$ [where $\langle \tilde{\xi}^2 \rangle = 2(1-\Gamma)k_B \tilde{T}$ and $\tilde{T} = \tau^2 T$], we can eliminate the period from the classical expressions, and define the new map

$$\bar{p} = \Gamma p + K[\sin x + a \sin(2x+\Phi)] + \tilde{\xi},$$

$$\bar{x} = x + \bar{p}. \quad (3)$$

In the quantum version of the model the system Hamiltonian is given by

$$\hat{H}_S = \hat{n}^2/2 + V(\hat{x},t), \quad (4)$$

where the quantization has been performed in such a way [17] that $x \to \hat{x}$, $n \to \hat{n} = -i(d/dx)$ and $\hbar = 1$. This amounts to saying that, being $[\hat{x},\hat{p}] = i\tau$, there is an effective Planck constant given by $\hbar_{\text{eff}} = \tau$. Then, the classical limit corresponds to $\hbar_{\text{eff}} \to 0$, while keeping $K = \hbar_{\text{eff}} k$ constant.

In order to incorporate dissipation and thermalization to the quantum map we consider the coupling of the system to a bath of noninteracting oscillators in thermal equilibrium. The degrees of freedom of the bath are eliminated introducing the usual weak coupling, Markov and rotating wave approximations [18]. This leads to a Lindblad equation in action representation for the density matrix of the system ρ,

$$\dot{\hat{\rho}} = -i[\hat{H}_S,\hat{\rho}] + g\sqrt{n^+_{th}(\Omega_n,T)n^+_{th}(\Omega_{n'},T)}\{[\hat{L}_n,\hat{\rho}\hat{L}^\dagger_{n'}] + [\hat{L}_n\hat{\rho},\hat{L}^\dagger_{n'}]\} + g\sqrt{n^-_{th}(\Omega_n,T)n^-_{th}(\Omega_{n'},T)}\{[\hat{L}^\dagger_n,\hat{\rho}\hat{L}_{n'}] + [\hat{L}^\dagger_n\hat{\rho},\hat{L}_{n'}]\}, \quad (5)$$

where the frequencies $\Omega_n = n + 1/2$ are the energy differences between two neighboring levels of the rotator. The population densities of the bath found in Eq. (5) are given by

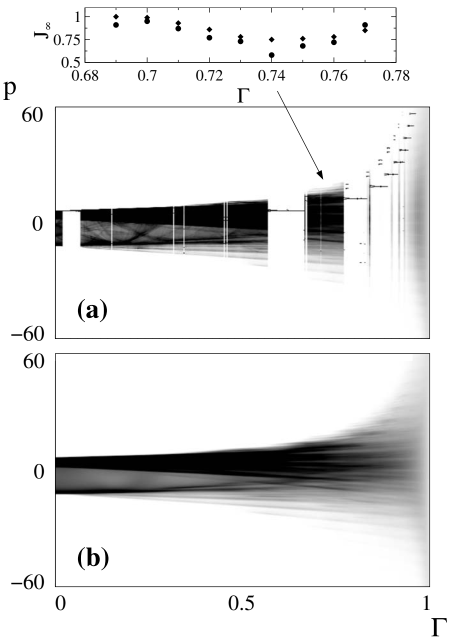

FIG. 1. Bifurcation diagrams in terms of p, as a function of the parameter Γ. We show the last 5×10^3 classical p values corresponding to 5×10^3 random initial conditions taken in the interval $(x\in[0,\pi), p\in[-\pi,\pi])$, and after 1.4×10^5 steps of the map. In panel (a) $T=0$ (on top, the asymptotic current J_∞ as a function of Γ for the indicated irregular window can be seen), while in (b) $T=0.05$.

$$n^-_{th}(\omega,T) = \frac{1}{\exp(\hbar\omega/k_B T) - 1},$$

$$n^+_{th}(\omega,T) = n^-_{th}(\omega,T) + 1. \quad (6)$$

The system operators $\hat{L}_n = \sqrt{|l|+1}(|n\rangle\langle n+1| + |-n\rangle\langle -n-1|)$ describe transitions towards the ground state of the free rotator. Requiring quantum to classical correspondence at short times we fix the coupling constant $g = -\ln(1-\Gamma)$. For $T=0$ we recover the master equation used in [15] for the pure dissipative case. For finite temperature, the last term in Eq. (5) describes the thermal excitation of the rotator through absorption of heat bath energy. Finally, Eq. (5) will be integrated numerically without further approximations.

III. RESULTS

Since we are interested in chaotic transport, throughout the following calculations we will use the set of parameters given by $K=0.7$, $\Phi=\pi/2$, and $a=0.7$. In the Hamiltonian limit, this case shows no visible stability islands in phase space. We have first studied some classical aspects of our system at zero temperature, beginning with the bifurcation diagrams in terms of p as a function of the parameter $\Gamma \in [0,1]$ [see Fig. 1(a)]. These diagrams show the projections of phase space on the p axis for all the possible values of dissipation. This allows us to determine which Γ correspond to chaotic (gray and black regions) or regular behavior (thin lines). It should be mentioned that the chaotic attractors set in very fast for small Γ. But this is not the case for larger values where the transient times can be very long. For this reason, we have calculated these diagrams with the last 5×10^3 iterations of the map, after the first 1.4×10^5 have been discarded. We have randomly taken 5×10^3 initial conditions inside the unit cell $(x \in [0,\pi), p \in [-\pi,\pi])$. Several regular and chaotic windows alternate. The former are characterized by simple attractors (stable fixed points of the dissipative map) and the latter are dominated by chaotic attracting sets. The width in p of these sets grows as dissipation weakens $(\Gamma \to 1)$.

The main quantity characterizing transport is the current $J(t)=\langle p_t \rangle$, where $\langle \cdots \rangle$ stands for the average taken on the initial conditions, and p_t is the moment after the tth iteration of the map. Since bifurcations that change the shape of strange attractors also play a role in determining the values of the asymptotic current we restrict our analysis to the chaotic window approximately located at $\Gamma \in [0.68, 0.78]$ indicated by an arrow. The inset of Fig. 1(a) displays the asymptotic current $J_\infty = \lim_{t\to\infty} J(t)$ as a function of Γ (circles stand for $T=0$). At this Γ range, 100 kicks are enough to reach the stationary behavior, independently of the initial distributions. As pointed out in [15], dissipation induces an asymmetry of the strange attractor which is responsible for the directed transport. On the other hand, this dissipation mechanism contracts phase space and makes the higher energies inaccessible for the system. The final value of $\langle p \rangle$ results from the interplay between both effects. In fact, increasing dissipation does not increase the transport and the largest values of $\langle p \rangle$ are obtained for the lowest values of dissipation, i.e., $\Gamma \gtrsim 0.9$ (for example, $J_\infty = 5.78$ for $\Gamma = 0.97$). The minimum current in this window is reached for an intermediate value of dissipation ($\Gamma = 0.74$).

We then consider the case of finite temperature. The bifurcation diagram corresponding to $T=0.05$ is shown in Fig. 1(b). It is clear that the effect of temperature consists of smoothing the finer structure of the chaotic attractors that is present for the smallest values of Γ. Even for this extremely low value of T the detailed features have almost completely disappeared, with the exception of the black lines corresponding to the highest values of the density distributions. The other very interesting effect is that temperature erases the regular windows allowing for a continuously chaotic behavior. This could be of much relevance in obtaining large ratchet currents without the need for an extremely fine-tuned, weak dissipation [19] (this will be addressed in future studies [20]). As shown in the inset, low temperatures (diamonds correspond to $T=0.05$) lead to a noticeable enhancement of the asymptotic current J_∞ (around 30% for $\Gamma = 0.74$).

We now turn to compare the classical and quantum behaviors. First, we analyze the currents [which in the quantum case is given by $J(t) = \mathrm{tr}(\hat{\rho}\hat{p})$]. In Fig. 2 we display J_∞ as a function of T, for three different values of Γ and $\hbar_{\mathrm{eff}}$. At the classical level low temperatures lead to an enhancement of the current for intermediate values of the dissipation (see Fig. 2 upper and middle panels corresponding to $\Gamma = 0.7, 0.75$). In the case of weak dissipation ($\Gamma = 0.9$ in the lower panel of Fig. 2), which displays larger values of J_∞, the effect of thermal noise is negligible. For higher temperatures, the thermal effects reduce $\langle p \rangle$ in all cases. This can be interpreted as follows: Thermal noise reduces the energy loss caused by dissipation (with no kicks, the system would attain a Boltzmann distribution), so higher energies can be reached, in comparison with a pure dissipative process. But since this diffusion also tends to blur the asymmetry of the strange attractor, the two effects compete and transport has a maximum for low values of T, and then decreases.

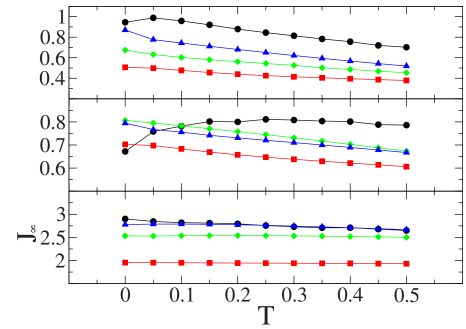

FIG. 2. (Color online) Asymptotic current value J_∞ as a function of temperature T. Upper panel shows the case for $\Gamma=0.7$, the middle one for $\Gamma=0.75$, and the lower one for $\Gamma=0.9$. Black circles stand for the classical values. The quantum cases correspond to $\hbar=0.055$ (blue triangles), $\hbar=0.165$ (green diamonds), and $\hbar=0.494$ (red squares). As initial conditions we have taken 10^6 random points (classical) and a density operator with equal population at all the possible p eigenstates inside the phase-space region given by $(x \in [0,\pi), p \in [-\pi,\pi])$.

At the quantum level we can clearly see that the previously described thermal enhancement is not generally present, at least for the $\hbar_{\mathrm{eff}}$ values we have considered. It is important to note that these values are consistent with experimental implementations using, for example, cold sodium atoms in a laser field having a wavelength $\lambda = 589$ nm (for more details see [9,15]). However, we observe a very slight growth of the current for the $\hbar_{\mathrm{eff}} = 0.055$ case with $\Gamma = 0.9$ (see the blue triangles in the lower panel of Fig. 2) indicating that the temperature dependence of the current is very sensitive to the particular dynamics of the system. For a different example, an enhancement of the quantum transport has been observed [14], hence it is difficult to extract universal behaviors. For $\Gamma = 0.7$ the quantum current is lower than the classical one for all temperatures, as already pointed out in [15], but only for $T=0$. The same happens in the case of weak damping $\Gamma = 0.9$ (nevertheless, for $\hbar_{\mathrm{eff}} = 0.055$ and $T \gtrsim 0.1$ both currents coincide). The case $\Gamma = 0.75$ shows a different behavior. At $T=0$ the quantum currents (for any of the $\hbar_{\mathrm{eff}}$ values we have considered) are larger than the classical ones, that is, there is an enhancement due to the finite size of $\hbar_{\mathrm{eff}}$. Also, the maximal quantum current corresponds to $\hbar_{\mathrm{eff}} = 0.165$, and this is valid for all the temperatures shown (for $T=0.05$ it is still greater than the classical one). For larger quantum coarse graining (see $\hbar_{\mathrm{eff}} = 0.494$) quantum currents decrease. In this sense, it seems that the effect of quantum fluctuations on the quantum directed currents is analogous to the effect of those of thermal origin on the classical ones. Small fluctuations of thermal or quantum mechanical origin assist directed transport while large fluctuations (corresponding to high temperatures or to large values of $\hbar_{\mathrm{eff}}$, respec-

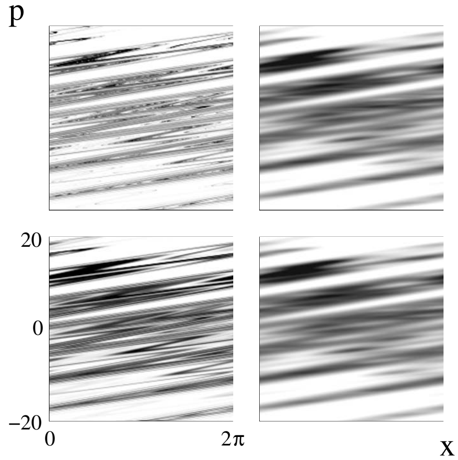

FIG. 3. Phase space portraits for $\Gamma=0.75$ at $t=40$. Left-hand panels correspond to the classical while the right-hand panels correspond to the quantum strange attractors. In the upper panels $T=0$, while in the lower ones $T=0.1$. As initial conditions we have taken 10^6 random points (classical) and a density operator with equal population at all the possible p eigenstates inside the phase-space region given by $(x\in[0,\pi), p\in[-\pi,\pi])$.

tively) blur the asymmetry of the attractor and thus kill the net current.

It is interesting to note that in the case $\Gamma=0.75$ the thermal diffusion associated with the temperature which gives the maximal value of the current ($<\xi^2>=0.12$ for $T=0.25$) is of the order of the quantum coarse graining $\hbar_{\rm eff}$ corresponding to the strongest quantum current. For $\Gamma=0.7$ the classical current attains its maximum value at $T=0.05$ ($<\xi^2>=0.03$), which corresponds to a value of $\hbar_{\rm eff}$ that we were not able to consider in our numerical calculations.

The analogy between thermal noise and quantum coarse graining can also be appreciated when looking at the asymptotic Poincare sections and Husimi distributions (displayed in Fig. 3 for $\Gamma=0.75$, $\hbar_{\rm eff}=0.055$). As expected, at zero temperature the quantum Husimi function reproduces well the main patterns of the classical attractor but shows less fine structure (see the upper panels). If a small temperature is introduced the fine details of the classical distribution are smoothed out and both distributions look more alike (see the lower panels corresponding to $T=0.1$). On the other hand the quantum distributions at zero and finite temperatures are practically indistinguishable, indicating that the quantum coarse graining is at least of the order of the thermal one for these values of T.

We finally study $J(t)$ as a function of t (i.e., the number of iterations of the map). Results for $\Gamma=0.75$ are shown in Fig. 4, where different temperatures and $\hbar_{\rm eff}$ values have been considered. Besides the mentioned fact that the asymptotic value is reached very rapidly, we notice that the transient behavior shows a very close quantum-to-classical correspondence. The classical current peak observed at $t\sim 10$ for low

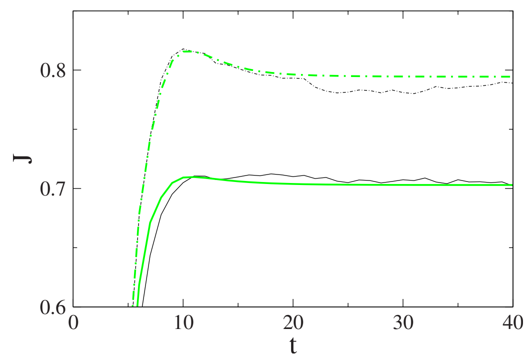

FIG. 4. (Color online) Current J as a function of time t, for coupling strength $\Gamma=0.75$. Thin black lines correspond to the classical values, while thick green (gray) lines correspond to the quantum cases (in these last cases we always take $T=0$). We show results for $T=0.1$ and $\hbar_{\rm eff}=0.055$ (dotted-dashed lines), and for $T=0.85$ and $\hbar_{\rm eff}=0.494$ (solid lines). As initial conditions we have taken 10^7 random points (classical) and a density operator with equal population at all the possible p eigenstates inside the phase-space region given by $(x\in[0,\pi), p\in[-\pi,\pi])$.

temperatures ($T=0.1$) is also present in the quantum current when $\hbar_{\rm eff}=0.055$ and $T=0$. This peak disappears from the classical current at larger temperatures ($T=0.85$), and so does the quantum one at $\hbar_{\rm eff}=0.494$ and $T=0$. So the analogy seems to hold at all times.

IV. CONCLUSIONS

In this work we have analyzed the behavior of a chaotic dissipative system that shows directed transport under the influence of a thermal bath, both in its classical and quantum versions. We have varied the strength of the coupling with the environment and also the temperature. We have found that the transport enhancement effect due to a finite temperature is highly dependent on the system specific properties. In fact, it depends on the coupling strength of the system with the environment and also on the $\hbar$ size. Moreover, we could find an analogy between the effects caused by thermal and quantum fluctuations. These results open the possibility for many further studies that include finding ways of obtaining large ratchet currents in experimentally realistic situations in kicked Bose-Einstein condensates (BECs) and cold atoms experiments. These are one of the best candidates to test our results since even BECs show an unavoidable fraction of noncondensed atoms when kicked. If kicks become strong, thermal excitations will be of much relevance rather than a negligible effect. With the sort of calculations presented in this paper the effects of this fraction on the transport properties of the system could be estimated.

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